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A SURVEY OF STRATEGIC MARKET GAMES

ABSTRACT: *The Strategic Market Game (SMG) is the general equilibrium mechanism of strategic reallocation of resources. It was suggested by Shapley and Shubik in a series of papers in the 70s and it is one of the fundamentals of contemporary monetary macroeconomics with endogenous demand for money. This survey highlights features of the SMG and some of the most important current applications of SMGs, especially for monetary macroeconomic analysis.*

KEY WORDS: *general equilibrium, strategic market game, macroeconomics, monetary economics*

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1. INTRODUCTION

The Strategic Market Game (SMG) is the general equilibrium mechanism of strategic reallocation of resources.[1] It was suggested by Shapley and Shubik (1977) and it is one of the fundamentals of monetary macroeconomics with endogenous demand for money. One can think about the SMG as a generalization of Cournot and Bertrand competition within a general equilibrium framework.

Applications of the SMG allow for elimination of the historical border between macro and microeconomics, i.e., to endogenize demand for money. This innovation is important for studying monetary policy and financial turbulence (Shubik, 2004; also in Goodhart, Sunirand and Tsomocos, 2006, and Tsomocos, 2003).

To be more specific, the approach offered in the seminal paper by Shubik and Wilson (1977) transforms the traditional macroeconomic problem of fiat money holding (Hahn paradox[2]) for a finite time, into a standard microeconomic problem.

The only existing survey of strategic market games was published by Giraud (2003), as an introduction to the specialized issue of the Journal of Mathematical Economics on SMGs in 2003. The current survey concentrates on some properties of the SMG, dispersed in literature, and some macroeconomic applications.

The survey has the following structure. First, we present a general equilibrium model of the SMG, some special properties of the SMG, and some traditional applications. Then we survey macroeconomic applications with money and some results of existing experiments.

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[1]   "One can think about SMG as a specially devised tool for studying strategic reallocation within general equilibrium theory with fiat money." (Shubik, p.10, (2004)).

[2]   Why to accept non-consumable money as a means of payment at the final moment of finite time trade, and by backward reasoning in any earlier periods.

2. CHARACTERIZATION OF MEANS OF PAYMENT IN SMGS

SMGs explicitly assume the existence of a specialized means of trade, which is paid for a consumable good. Quint and Shubik (2004) describe the following types of goods that can serve this purpose.

1. Perishable, which have a single period of life, when they are traded and are consumed.
2. Storable consumable, for example, cans of beans, spices, or salt. They can be consumed, "at the option of an individual" or be carried over in time.
3. Durable good, which supplies a stream of services during its finite lifetime.
4. Fiat money, "a fictitious durable" with no consumption value, but with value "derived from its participation and usage in transactions" (Kiyotaki and Wrigh, 1989).

The key difference between fiat money and any consumable good is that, for any period in a multi-period trade, money is a stock variable accumulated in the economy, and all other goods are flow variables, which loose their values.

Existence of a special means of payment has a direct effect on the structure of the market system and on the number of markets in the economy. (See subsection "Enough money").

2.1 Characterization of an economy

An economy is characterized by:

1. A set of players (traders), I, with a general element i. Each trader is characterized by an initial endowment, $e^i, e^i \in R^m$, where L is the number of consumable goods, and $m = L + 1$ is the total number of goods in the economy, both consumable and money.
2. A message (a signal) is a statement of a trader, which includes his/her decisions of what, how, and under which conditions to buy *or* to sell or to buy *and* to sell. A message is the true information about a consequent strategic action of the player. In this survey the strategies of players are their market messages.

3. A market (a trading post) processes messages and produces trades, i.e., collects messages of traders, determines prices, performs settlements, payments, and delivery procedures, following some predefined rules.

2.2 Market organization

Shubik (2004) described the different organization of markets in economies using SMGs. However, there are some common features.

Each good has at least one trading post, which collects all messages concerning this good from the traders. This trading post is responsible for all operations with this good (like the Walrasian auctioneer). But the situation is more delicate than in the Walrasian approach. If there is a trading post for every pair of commodities, then there are $L(L-1)/2$ different markets. If each trading post for each good $l \in \{1,...,L\}$ trades only for money, the good with the number $m = L+1$, then there are only L markets. This detail matters for SMGs with fiat money and credits. Koutsogeras (2003) presented some important results for multiple post trading (see further below).

The organization of markets matters for trade with endogenously defined prices (Shubik, 1976). This is not essential in a centralized trade of Walras.

Capie, Tsomocos and Wood (2006) used the SMG to analyse conditions under which fiat money dominates electronic barter and studied the implications for monetary policy, due to technological innovations in finance. Shubik's experiments demonstrated that the organization of markets also matters for an equilibrium transition path (see the section on experiments with SMGs).

2.3 Variety of strategies in SMGs

Strategic market games supply multiple options for constructing trading strategies. The options differ by the dimensions of individual strategy sets. SMGs permit several trading mechanisms (Shubik and Quint, 2004):

1. The sell all model.
2. The buy-sell model with pure strategies (pure strategies wash sales model).

3. The double auction model.
4. The mixed strategies model, when wash sales are allowed (see further).

2.4 Sell all model

This model assumes that a trader sells the total quantity of the good s/he has and pays with his/her money for what s/he buys. This model was studied, for example, by Shubik (1959).

2.5 Wash sales

A wash sale is a set of a trader's simultaneous buy and sell operations in the same good and at the same trading post. Quantities bought/sold are bounded by individual endowments, availability of credits, and individual budget constraints. The dimension of a typical strategy set is $2L$. Properties of this trading mechanism are discussed further below.

The motivations for using wash sells are strategic signalling reasons, different bookkeeping, and tax-reduction interests. In some countries for some markets wash sales are illegal and/or prohibited.

Peck and Shell (1990, 1992) studied liquidity of the market with wash sales and proved the existence of multiple-trading equilibriums for this game. Ray (2001a) studied how individually achievable allocations depend on the total bid and the total offer made by other traders. He demonstrated that trade with wash sales and trade with buy or sell strategies are not individually decision equivalent (Ray, 2001b) and have different sets of achievable allocations.

3. THE BASIC SMG AND ITS PROPERTIES

3.1 The buy-sell model with commodity money

A consumable good is labelled as l, $l \in \{1,...,L\}$, where L is the number of different types of consumable commodity. There is a selected consumable good which serves as commodity money, i.e., a means of payment, labelled $L+1$.

Each trader i from I has a preference relation, represented by a utility function $u^i : R_+^m \to R$, strictly concave and differentiable in R_+^m, $m = L+1$. Before the game each trader receives an endowment of commodities and an endowment of commodity money, $e^i \in R_+^m, e^i \neq 0, \sum_{i \in I} e_l^i > 0$ for every good $l, l \in \{1, ..., L\}$.

$E = \left(R_+^m, \left(u^i, e^i \right) \right)_{i \in I}$ is an economy.

Every commodity $l, l \in \{1, ..., L\}$ is only traded at one post. Traders can forward their bid and offer this good to this post. Trade takes place only for fiat money, which reduces the number of markets and prices. The consequences of multi-post trading are presented further on.

Every trader i has a strategy set which describes all his bids and offers for every commodity:

$$S^i = \{s^i = (b^i, q^i) \in R^{2L} : 0 \leq q_l^i \leq e_l^i; 0 \leq \sum_{l=1,L} b_l^i \leq e_{L+1}^i, l = 1, ..., L\}$$

with a general element $s^i = (s_l^i)_{l=1,L}$, where $s_l^i = (b_l^i, q_l^i)$ - is a pair, which consists of a bid, b_l^i, nominated in the good $m = L+1$ (the means of payment), and an offer, q_l^i, nominated in terms of a commodity l.

Let $S^{-i} = \prod_{n \neq i} S^n$ be the set of strategies of all other players besides i with a general element s^{-i}, S^{-i} is a non-empty convex compact in $R_+^{2L(I-1)}$, where I is a number of traders. The set of strategies in the game $S = \prod_{n \in I} S^n$ is a non-empty convex compact in R_+^{2LI}.

3.2 Pricing

Each good has one trading post, where it is traded for commodity money. All traders send their messages about the good to the trading post, where it is traded.

An aggregated bid for a commodity l is $B_l = \sum_{n \in I} b_l^n$ and an aggregated offer is $Q_l = \sum_{n \in I} q_l^n$. Every trader i meets an aggregated bid from his/her competitors $B_l^{-i} = \sum_{n \neq i} b_l^n$ for the commodity l and an aggregated offer from his/her competitors $Q_l^{-i} = \sum_{n \neq i} q_l^n$ for the same commodity.

The pricing mechanism for a good $l \in \{1, ..., L\}$ is defined as

$$p(b_l^i, q_l^i, B_l^{-i}, Q_l^{-i}) = \frac{b_l^i + B_l^{-i}}{q_l^i + Q_l^{-i}} \text{ if } q_l^i + Q_l^{-i} \neq 0$$

and

$$p(b_l^i, q_l^i, B_l^{-i}, Q_l^{-i}) = 0 \text{ if } q_l^i + Q_l^{-i} = 0.$$

For a market with a finite number of traders, each trader has some market power to affect the final price. However, if there is finite number of type of traders, and every type is represented by a continuum of traders, then the solution of the game is competitive as the individual market power of a trader is negligible.

3.3 Allocation rules

After-trade allocation for every trader $i \in I$ for a commodity $l \in \{1, ..., L\}$, if $q_l^i + Q_l^{-i} \neq 0$ follows the rule:

$$x_l^i = e_l^i - q_l^i + \frac{b_l^i}{p_l}, p_l = \frac{b_l^i + B_l^{-i}}{q_l^i + Q_l^{-i}}$$

If $i \in I$ does not violate his/her budget constraint, i.e.,

$$p_l > 0, q_l^i > 0, \sum_{l=1,L} p_l q_l^i \leq \sum_{l=1,L} b_l^i,$$

and

$$x_l^i = e_l^i - q_l^i$$

if the budget constraint is violated, i.e., $p_l > 0, q_l^i > 0, \sum_{l=1,L} p_l q_l^i > \sum_{l=1,L} b_l^i$.

Naturally we define $x_l^i = e_l^i$ if $q^i = 0; b^i = 0$.

Condition $\sum_{l=1,L} p_l q_l^i \leq \sum_{l=1,L} b_l^i$ means that trader $i \in I$ does not violate his/her budget constraint (given endogenous prices in the economy). Otherwise s/he is punished by confiscation of all his/her offers.

Possible insolvency in payments, $\sum_{l=1,L} p_l q_l^i > \sum_{l=1,L} b_l^i$, (given endogenous prices), has a very important role in monetary applications and will be discussed further.

Allocation of the commodity money $m = L + 1$ for any trader $i \in I$ follows the rule:

$$x_i^m = e_i^m - \sum_{1,L} b_l^i + \sum_{1,L} q_l^i p_l$$

if i does not violate budget constraint, $\sum_{1,L} b_l^i \geq \sum_{1,L} q_l^i p_l$ and $x_i^m = e_i^m - \sum_{1,L} b_l^i$, if the budget constraint is violated, $\sum_{1,L} b_l^i < \sum_{1,L} q_l^i p_l$. Similarly $x_i^m = e_i^m$ if $q_1^i = 0$, $b_1^i = 0$.

3.4 Utility and maximization problem

Every trader $i \in I$ has a utility maximization problem:

$$\max_{(b_l^i, q_l^i)_{l=1}^L \in S^i} u^i \left(x_l^i (b_l^i, q_l^i, B_l^{-i}, Q_l^{-i})_{l=1}^L \right)$$

subject to the pricing and allocation rules defined above. A trader maximizes his utility from the trade by affecting market prices.

A SMG is the game $\left(I, \left(S^i, u^i \right)_{i \in I} \right)$ for strategic reallocation of resources within the general equilibrium framework. Thus strategic market games have properties of a resource reallocation mechanism and properties of a non-cooperative game.

Nash equilibrium in pure strategies is a profile of strategies $\left(s^{i*} \right)_{i \in I}$ such that

$$u^i \left(x^i \left(s^{i*}, s^{-i*} \right) \right) \geq u^i \left(x^i \left(s^i, s^{-i*} \right) \right)$$

for every player (trader) $i \in I$ and $s^i \neq s^{i*}$.

4. PROPERTIES OF SMGS

4.1 Existence

A trivial equilibrium, i.e., an equilibrium when all traders use trivial strategies or do not trade, is always equilibrium in the game.

If a trader uses any trading strategy, then the utility function has discontinuity at origin, which complicates analysis of the game. In order to avoid this Dubey and Shubik (1978) studied the existence of ε-Nash equilibrium. Their idea is to assume that there is an outside agency, which supplies small $\varepsilon > 0$ amounts of each good to each trading post, so that the price is defined as:

$$p(b_l^i, q_l^i, B_l^{-i}, Q_l^{-i}) = \frac{b_l^i + B_l^{-i} + \varepsilon}{q_l^i + Q_l^{-i} + \varepsilon}.$$

Dubey and Shubik (1978) proved that for any small and positive ε the price is positive and bounded from above. They proved that for this condition the game has an internal Nash equilibrium.

A SMG with wash sales has multiple trading equilibriums (Hubert and Shubik, 2009). Peck, Shell and Spear (1992) showed that if there is one trading equilibrium for a finite number of traders, then there are other trading equilibriums.

4.2 Efficiency

Dubey (1980) demonstrated that usually SMGs with a finite number of players are inefficient, as each player does his best to exploit individual market power to affect market price. Asymptotic efficiency of SMGs was studied by Dubey, Mas-Colell and Shubik (1980). For some standard assumptions they proved that for a small number of traders a result of the SMG is inefficient, but it has an asymptotic convergence to the efficient Walrasian outcome when the number of traders increases infinitely. Then individual market power becomes negligible.

Koutsougeras (2009) studied the degree of competition in SMGs with a finite number of traders. He explicitly demonstrated a decrease in individual market power: "the proportion of individuals whose strategic behaviour differs substantially from price taking, converges to zero" asymptotically, "regardless of the distribution of characteristics" of the players. Koutsougeras and Ziros (2007) demonstrated that for an atomless economy there is a three-way equivalence between market game mechanisms, competitive equilibriums, and a core.

Dubey and Geanakoplos (2003) constructed the Walrasian equilibrium as an asymptotic case of non-cooperative games using a variant of the SMG. Amir and Bloch (2009) reproduced the convergence of the SMG with wash sales to competitive equilibrium from super modular optimization/games, using asymptotic replication of an economy.

4.3 Trivial strategy

A trivial strategy is a strategy where a trader's bids and offers are zero. A trivial strategy in SMGs leads to no-trade equilibrium. If the trivial equilibrium is the only equilibrium in the game, then the initial allocation of endowments is already Pareto efficient and the game does not have any trading equilibrium.

A SMG with wash sales has multiple trading equilibriums (Hubert and Shubik, 2009). Peck, Shell and Spear (1992) showed that if there is one trading equilibrium for a finite number of traders, then there are other trading equilibriums. From another perspective, Busetto and Codognato (2006) showed that prohibition of wash sales may lead to the situation where only trivial strategies are rational.

Indeterminacy with multiple equilibrium can be overcome by using mixed strategies for SMGs with wash sales, introduced in Levando, Boulatov, Tsomocos, (2012) (see further on). A mixed strategies approach eliminates the problem of trivial equilibrium, as trivial strategies are assigned zero probability (if the game has some non-trivial trading equilibrium).

4.4 Time and retrading

Ghosal and Morelli (2004) studied dynamic retrading for SMGs with a sell all strategy. They demonstrated Pareto improvement in allocations and time-asymptotic competitiveness. This result matches replication of a one-period game with sell all strategies, which asymptotically converges to a competitive outcome in a one period game (Shubik, 2004).

Studying dynamic trade in SMGs with wash sales is complicated by possible equilibria multiplicity in a one period game.

4.5 Special properties of SMGs with wash sales

SMGs with wash sales have some special properties.

Identity strategy

We can ask a question: are there strategies that do not change allocation for a player i after the trade (presented, for example, in Dubey and Shubik, 1977)?

These strategies must satisfy the following conditions:

$$e_l^i = e_l^i - q_l^i + b_l^i \frac{q_l^i + Q_l^{-i}}{b_l^i + B_l^{-i}} \qquad \text{for} \qquad \text{consumable} \qquad \text{goods} \qquad \text{and}$$

$$e_m^i = e_m^i - \sum_{l=1,L} b_l^i + \sum_{l=1,L} q_l^i \frac{b_l^i + B_l^{-i}}{q_l^i + Q_l^{-i}} \text{ for money. From here it follows that:}$$

$$Q_l^{-i} b_l^i = B_l^{-i} q_l^i.$$

This means that there exists a strategy $\left(b_l^i, q_l^i\right)_{i=1,L}$, which maps the initial endowment $e^i = \left(e_1^i, ..., e_L^i, e_m^i\right)$ of player i into itself, given the strategies of other players.

Concavity of a set of individually achievable allocations

The set of individually achievable allocations $G\left(q_1^i, ..., q_{L+1}^i\right)$ serves as a budget constraint in the game. It is a strictly concave curve (Dubey and Shubik, 1977), but converges to a straight line with an asymptotic increase in the number of traders.

Let $G\left(q_1^i, ..., q_{L+1}^i\right) = e_{L+1}^i + \left(\frac{\left(e_l^i - q_l^i\right) B_l^{-i}}{Q_l^{-i} + e_l^i - q_l^i}\right)_{l=1}^{L}$ be a point on this line. One can

show that partial derivatives are $\partial G / \partial q_l^i > 0$ and $\partial G^2 / \partial^2 q_l^i < 0$ for $l = 1, L$.

Best response as a correspondence

SMGs with wash sales have the special property that the best response of a trader can be a non-trivial correspondence, i.e., may not be unique. We demonstrate this property below with a numerical example.

The example is a SMG with two goods and two players, i and j. We substitute the pricing and the allocations rules into the utility function. Trader i has a utility maximization problem:

$$\max_{s^i \in S^i} u^i \left(e_1^i - q_1^i + b_1^i \frac{q_1^i + q_1^j}{b_1^i + b_1^j}, e_2^i - b_1^i + q_1^i \frac{b_1^i + b_1^j}{q_1^i + q_1^j} \right)$$

The first order condition over the strategy q_1^i is

$$u_1^i * \left(-1 + \frac{b_1^i}{b_1^i + b_1^j} \right) + u_2^i * \left(\frac{b_1^i + b_1^j}{q_1^i + q_1^j} - q_1^i \frac{b_1^i + b_1^j}{\left(q_1^i + q_1^j \right)^2} \right) = 0$$

and for the strategy b_1^i is

$$u_1^i * \left(\frac{q_1^i + q_1^j}{b_1^i + b_1^j} - b_1^i \frac{q_1^i + q_1^j}{\left(b_1^i + b_1^j \right)^2} \right) + u_2^i * \left(-1 + \frac{q_1^i}{q_1^i + q_1^j} \right) = 0,$$

where $u_1^i = \partial u^i / \partial q_1^i$, $u_2^i = \partial u^i / \partial q_2^i$, which implies

$$\frac{1}{p} \frac{u_1^i}{u_2^i} = \frac{1 - \dfrac{q_1^i}{q_1^i + q_1^j}}{1 - \dfrac{b_1^i}{b_1^i + b_1^j}}.$$

From here we obtain that trader i has only one equation to determine two strategic variables q_1^i and b_1^i. This means that, if the SMG has a wash sale, then traders have a free choice of some components of their equilibrium strategies. This property of SMGs with wash sales leads to a multiplicity of trading equilibriums.

We can demonstrate best response correspondence with an example (from Levando, Boulatov and Tsomocos, 2012, further LBT). Let there be a SMG of two players where payoff function for $i = 1, 2$

$$
\Pi_i\left(s_i s_{-i}\right) = \log\left(e_1^i + b_1^i\left(q_1^i, q_1^{-i}\right) \frac{q_1^i + q_1^{-i}}{b_1^i\left(q_1^i, q_1^{-i}\right) + b_1^{-i}\left(q_1^i, q_1^{-i}\right)} - q_1^i \right) +
$$

$$
\log\left(e_2^i - b_1^i\left(q_1^i, q_1^{-i}\right) + q_1^i \frac{b_1^i\left(q_1^i, q_1^{-i}\right) + b_1^{-i}\left(q_1^i, q_1^{-i}\right)}{q_1^i + q_1^{-i}} \right)
$$

$e^1 = (10, 30)$ and $e^2 = (30, 10)$ are the endowments of the two traders. Figure 1 demonstrates the best responses of Trader 1 to two different strategies of Trader 2: the solid line is the best response to the strategy $\left(q_1^2, b_1^2\right) = (5, 5)$ and the dashed line is the best response to the strategy $\left(q_1^2, b_1^2\right) = (6, 20)$.

Figure 1. Best response correspondence of trader 1 to the two sets of strategies of trader 2.
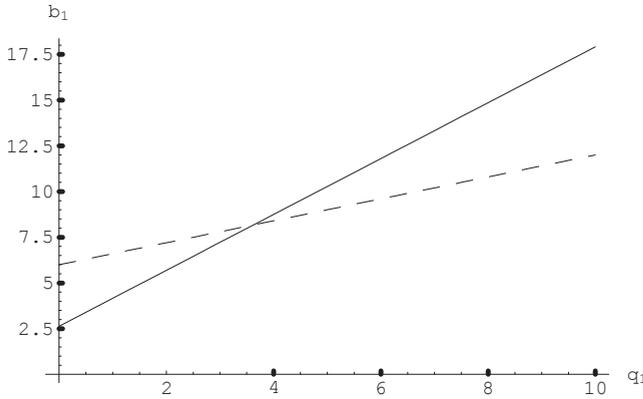
Source: Levando, Boulatov and Tsomocos, 2012.

Thus, if wash sales are permitted in a SMG, then a trader has a continuum of different strategies, with the same market price.

SMGs have another property, connected to that presented above. Shapley and Shubik (1977, p. 964) wrote: "... if at equilibrium Trader i is sending both goods and cash to the same trading post and if the price there is p, then he might consider decreasing both q^i and b^i in the ratio $1/p$. This would not change his final outcome (...) or the price, but it would change the marginal cost of good to the other traders and so destroy the equilibrium".

It is easy to show this formally. Consider, for simplicity, the game of two traders. The price p can be written down in two different ways:

$$p = \frac{\overline{b}_1^i + b_1^j}{\overline{q}_1^i + q_1^j} = \frac{b_1^i + b_1^j}{q_1^i + q_1^j},$$

where $\overline{b}_1^i, \overline{q}_1^i$ and $b_1^i q_1^i$ are the two different sets of strategies of player i.

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Then if $\bar{b}_1^i = \Delta b_1^i$ and $\bar{q}_1^i = \Delta q_1^i$ we can obtain that

$$\Delta b_1^i \left(q_1^i + q_1^j \right) = \Delta q_1^i \left(b_1^i + b_1^j \right) \text{ and } \frac{\Delta b_1^i}{\Delta q_1^i} = \frac{b_1^i + b_1^j}{q_1^i + q_1^j} = p,$$

what means if player i changes his/her strategies in the proportion of $1/p$, then the market price does not change.

Now we will demonstrate that this change in strategies does not change the allocation for player i. Let x_1^i be an allocation of good 1 for trader i:

$$x_1^i \left(q_1^i + \Delta q_1^i, b_1^i + \Delta b_1^i, q_1^j, b_1^j \right) = e_1^i - \left(q_1^i + \Delta q_1^i \right) + \left(b_1^i + \Delta b_1^i \right) \frac{q_1^i + q_1^j + \Delta q_1^i}{b_1^i + b_1^j + \Delta b_1^i} =$$

$$e_1^i - \left(q_1^i + \Delta q_1^i \right) + \left(b_1^i + \Delta b_1^i \right) \frac{1}{p} = e_1^i - q_1^i + \frac{b_1^i}{p^i} = x_1^i \left(q_1^i + \Delta q_1^i, b_1^i + \Delta b_1^i, q_1^j, b_1^j \right)$$

The same can be shown for the allocation of commodity money. Thus if i changes his/her strategy from $\left(q_1^i, b_1^i \right)$ to $\left(q_1^i + \Delta q_1^i, b_1^i + \Delta b_1^i \right)$ then s/he does not change his/her allocation and consequently his/her final payoff.

But from another perspective a change in the strategy of i results in a change of the ratio of marginal utilities for j:

$$\frac{u_1^i \left(q_1^i, b_1^i, q_1^j, b_1^j \right)}{u_2^i \left(q_1^i, b_1^i, q_1^j, b_1^j \right)} = \frac{\dfrac{b_1^i + b_1^j}{q_1^i + q_1^j} - q_1^j \dfrac{b_1^i + b_1^j}{\left(q_1^i + q_1^j \right)^2}}{\left(-1 + \dfrac{b_1^i}{b_1^i + b_1^j} \right)}$$

$$\neq \frac{\dfrac{b_1^i + \Delta b_1^i + b_1^j}{q_1^i + \Delta q_1^i + q_1^j} - q_1^j \dfrac{b_1^i + \Delta b_1^i + b_1^j}{\left(q_1^i + \Delta q_1^i + q_1^j\right)^2}}{\left(-1 + \dfrac{b_1^i}{b_1^i + \Delta b_1^i + b_1^j}\right)} = \frac{u_1^j\left(q_1^j + \Delta q_1^i, b_1^j + \Delta b_1^i, q_1^i, b_1^i\right)}{u_2^j\left(q_1^j + \Delta q_1^i, b_1^j + \Delta b_1^i, q_1^i, b_1^i\right)}$$

This means that, if there are wash sales, then traders have a free choice of some components of their equilibrium strategies, which implies indeterminacy in the game. To overcome this we need to use mixed strategies and an epistemic game theory approach.

4.6 Mixed strategies in SMGs

Indeterminacy in strategies makes players ask the question of how to overcome it using mixed strategies and how to construct them numerically. Mixed strategies are based on the conjectures of one player about the actions of another. This approach is studied in epistemic game theory (for example, Brandenburger, 2008). The criterion for selection of conjectures (or equilibrium probability functions) is the undominance of expected utility.

Levando, Boulatov and Tsomocos, (2012), introduced mixed strategies for SMGs. The construction of mixed strategies requires finding a probability function for each player that maximizes the expected payoff for a player, given the mixed strategies of the other. In other words, the equilibrium probability function generates an expected payoff not less than the expected payoff from any other probability function, given the equilibrium probability functions of all other players.

Let

$$G_i(\pi_i, \pi_{-i}) = \int_{\left(q_1^i, q_1^{-i}\right) \in [0, e_1^i] \times [0, e_1^{-i}]} \pi_i\left(q_1^i\right) \pi_{-i}\left(q_1^{-i}\right) \Pi_i(s) \, dq_i \, dq_{-i}$$

be expected utility of player i with normalization condition $\int_0^{e_i} \pi_i(q_i)\,dq_i = 1$, defined over a set of feasible offers $q_i \in [0, e_i]$ of player i. $\pi_i(q_i)$ is the probability that i chooses strategy q_i.

Variables q_i and q_{-i} are independent in the payoff function $\Pi_i(s)$, strategies $b_i^*(q_i, q_{-i})$ and $b_{-i}^*(q_{-i}, q_i)$ are determined with standard optimization procedures, so in an equilibrium payoff function $\Pi_i(s) = \Pi_i\big(q_i, b_i^*(q_{-i}, q_i), q_{-i}, b_{-i}^*(q_{-i}, q_i)\big)$ is constructed over the envelope surface.

$\left(\pi_1^*, \pi_2^*\right)$ is a profile of mixed strategies in the game if

$$G_i\left(\pi_i^*, \pi_{-i}^*\right) \geq G_i\left(\pi_i, \pi_{-i}^*\right), \ \forall \pi_i \neq \pi_i^* \text{ for } i \in \{1, 2\}$$

In order to construct mixed strategies, we need Euler-Lagrange equations for both traders. For trader $i, i = 1, 2$ it takes the form:

$$L_i(\pi_i) = \int_{(q_1^i, q_1^{-i}) \in [0, e_1^i] \times [0, e_1^{-i}]} \pi_i(q_1^i)\pi_{-i}(q_1^{-i})\Pi_i(s)\,dq_i\,dq_{-i} - \lambda_i \int_{(q_1^i) \in [0, e_1^i]} \pi_i(q_1^i)\,dq_i$$

where $\pi_{-i}^*(q_{-i})$ is the optimal mixed strategy of player $-i$ (not i) corresponding to the quantity q_{-i} of good 1 offered by player $-i$ to the market, and λ_i is the Lagrangian multiplier of player i. The mixed strategies are the two functions $\pi_i^* : [0, e_1^i] \rightarrow [0, 1]$ satisfying the normalization condition $\int_0^{e_i} \pi_i(q_i)\,dq_i = 1$.

Mixed strategies can be calculated numerically from the system of first order conditions, i.e.,

$$\text{for } i = 1, 2 : \lambda_i = \int_0^{e_1^i} \pi_{-i}\,(q_{-i})\Pi_i(s_i, s_{-i})\,dq_{-i}$$

These are Fredholm integral equations of the first type, which are usually ill-posed. One requires special numerical methods to solve them, for example, the regularization of Tikhonov. Details on existence, uniqueness, and numerical calculation of mixed strategies are in Levando, Boulatov and Tsomocos (2012).

4.7 Other applications of SMGs with consumable money

Production in strategic market games

At the moment there is very little literature on production and related issues within the framework of the SMG. Production in SMGs was introduced by Dubey and Shubik (1977). In their model strategic behaviour covers factor and final goods markets, which operate sequentially. In another paper Dubey and Shubik (1978) introduced the analysis of the actions of stockholders and managers into an economy with production and strategic reallocation of resources within a SMG trading mechanism.

Strategic market games and incomplete markets without money

Giraud and Weyers (2004) demonstrated the existence of sub-game-perfect equilibria for finite-horizon economies with incomplete markets without default. In their finite horizon model with strategic investors, the price of a security may be different from its fundamental value even if asset markets are complete, and regardless of the (finite) number of agents.

Brangewitz (2010) extended the model of Giraud and Weyers for cases with possibility of default with collaterals.

Transaction costs

Rogatsky and Shubik (1986) studied transaction cost effects on trading activity. They demonstrated that introduction of fiat money, which reduces transaction costs, also reduces the number of markets from $\frac{m(m-1)}{2}$ to $L = m - 1$, where m is the total number of goods in the economy, including commodity money, and L is the number of consumable goods.

Information and uncertainty in SMG

Dubey and Shubik (1977) presented the basic framework for SMGs when traders have asymmetric information. The investigation studied different asset market structures with exogenous uncertainty. Their model has asymptotic convergence to Arrow-Debreu markets. Peck and Shell (1989) showed that the Arrow securities game and the contingent commodity games have different Nash equilibriums for a finite number of players. Two games differ as the market power of a trader depends on market organization. The only common equilibrium between the games is one that does not use transfer of income across states (also in Weyers, 2002). Peck and Shell (1989) concluded that imperfectly competitive economies are sensitive to details of market structure, which are insignificant for competitive economies.

Goenka (2003) examined the effect of leakage of information in SMGs through prices. He demonstrated three results: (a) if information is free, then information revelation is faster; (b) if information is not free, then there may be no acquisition of information; (c) information leakage leads to a decrease in the value of information but does not affect the incentive for informed traders to sell the information.

Gottardi and Serrano (2005) studied a strategic model of dynamic trading with asymmetric information between buyers and sellers. The structures of the sets of buyers and sellers in their paper are different - a continuum of buyers and a finite number of sellers, who have private information. Information revelation in the trade significantly depends on the possibility for a seller to exploit his information, the presence of clients, the structure of the sellers' information, and the intensity of competition allowed by the existing trading rules.

Minelli and Meier (2011) proved the existence of an equilibrium in strategic market games for a large, anonymous market, in which both buyers and sellers may have private information. Polemarchakis and Raj (2006) studied correlated equilibrium in strategic market games played in an overlapping generation framework. They showed that it corresponds to sunspot equilibrium in the associated competitive economy.

Dubey, Geanakoplos and Shubik (1987) used a two period strategic market game to criticize the approach of Rational Expectations Equilibrium to asymmetric information in general equilibrium, as it does not leave room for private information to enter the market.

Asset trading

Koutsougeras and Papadopoulos (2004) studied saving behaviour. They demonstrated that in equilibrium with a finite number of traders there is a positive spread between the cost of a portfolio and the portfolio's returns, i.e., net profit for portfolio holders. Hens et al. (2004) constructed a two-fund separated strategic market game, where traders use sell-all strategies. Giraud and Stahn (2008) studied a two-period financial economy and addressed the question of the existence of an equilibrium. They showed the existence of nice equilibrium, i.e., a situation in which prices for both assets and commodities are strictly positive.

Equilibrium deviations from parities in prices and multiple post trading

Multiple-post trading is a trade where for some commodities there is more than one trading post. Such a trade may have equilibrium deviations from price parities.

Amir et al (1990) studied an economy with pair-wise trade in each good. They demonstrated that equilibrium prices may not satisfy parities for a finite number of traders. A similar argument was supplied by Sorin (1996) for studying SMGs with multiple fiat money.

Koutsougeras (1999, 2003) continued to study equilibrium deviations from price parities for multiple-post trading. He demonstrated that if a good can be traded at multiple trading posts, then equilibrium prices for the same good at different trading posts can be different. This leaves room for free arbitrage for newcomers.

Let k_1, k_2 be two different trading posts for the same good l. B_{l,k_1}^{-i} is an aggregate bid of all other traders besides i for good l trading at the trading

post k_1. B_{l,k_2}^{-i} is defined in the same way. Q_{l,k_1}^{-i} is an aggregate offer of all other traders besides i for good l trading at the trading post k_1. Q_{l,k_2}^{-i} is defined in the same way.

Equilibrium prices at every pair of trading posts k_1, k_2 of a commodity l are connected by the following (no-arbitrage) conditions

$$\left(p_{l,k_1}\right)^2 = \frac{B_{l,k_1}^{-i}}{Q_{l,k_1}^{-i}} \frac{Q_{l,k_2}^{-i}}{B_{l,k_2}^{-i}} \left(p_{l,k_2}\right)^2,$$

where there are no liquidity constraints, i.e., where for each player the value of all offers is less than the value of all bids. The proof follows from the first order conditions of utility maximization.

If some traders have binding liquidity constraints, then prices at different posts are connected as $\left(p_{l,k_1}\right)^2 \leq \dfrac{B_{l,k_1}^{-i}}{Q_{l,k_1}^{-i}} \dfrac{Q_{l,k_2}^{-i}}{B_{l,k_2}^{-i}} \left(p_{l,k_2}\right)^2$.

Gobillard (2005) and Bloch and Ferret (2001) demonstrated that the existence of wash sales is the necessary condition for the existence of this effect.

This result adds the argument for parity differentials being due only to oligopolistic multi-market trade. Koutsougeras and Papadopoulos (2003) and Papadopoulos (2008) constructed applications of this mechanism for international economics; for interest rate parities, purchasing power parities, and international Fisher equations.

5. DOUBLE AUCTION

This variant of SMG can be considered as the two-sided Bertrand-Edgeworth model (Dubey and Shubik, 1980). Traders can sell and buy goods by sending prices for buying/selling orders and by sending quantity limits for execution of transactions for these prices. The dimension of an individual strategy set is $4L$,

where L is the number of consumable goods. Mertens (2003) made an extensive investigation of the limited order market. In this section of the survey we follow the notation of Dubey and Shubik (1980).

An economy E consists of a set of traders $N = \{1, ..., n\}$, with a trader indexed by i. Every trader i is characterized by an initial endowment $a^i \in R_+^k$ and a utility function $u^i : R_+^k \to R$, where k is the number of goods. The utility function is assumed to be continuous, non-decreasing, and strictly increasing in at least one variable. In this survey we skip the properties of allocation of the endowments, which serve to guarantee the existence of the trade.

For any price p, $p \in R_{++}^k$ let the budget set of a trader i be $B^i(p) = \{x \in R_+^k : px \leq pa^i\}$ and let $\tilde{B}^i(p) = \{x \in B^i : u^i(x) = \max_{y \in B^i(p)} u^i(y)\}$.

A strategy for player $i \in N$ is a list $s^i = \left(p^i, q^i, \tilde{p}^i, \tilde{q}^i\right)$, where $p_j \in R_{++}^k$, $q^i \in R_{++}^k, \tilde{p}^i \in R_{++}^k, \tilde{q}^i \in R_{++}^k, \tilde{q}_j^i \leq a_j^i$, for every good $j = 1, k$. Elements of the strategy have the following interpretation: "if the price of commodity j is p_j or less, then i is willing to buy this good up to the quantity q_j^i; if the price of good j is $\tilde{p}_j$ or more, then trader i is willing to sell this good up to the quantity $\tilde{q}_j^i$".

Competitive equilibrium of the economy is a list $\left(p; x^1, ..., x^n\right)$ of prices and allocations such that each player i maximizes his utility in the set budget, $x^i \in \tilde{B}^i(p)$ and there is the balance of all goods in the economy $\sum_{i \in N} x^i = \sum_{i \in N} a^i$.

Dubey (2009) showed that the double auction mechanism yields competitive (Walras) allocations and strategic (Nash) equilibrium, even if there is a bilateral monopoly. Dubey and Sahi (2003) studied the mechanism of mapping signals into trade, which satisfies certain axioms and shows that there are only a finite number of such mechanisms, which satisfy these axioms. They also indicated

that there are some open problems regarding the convexity property of these mechanisms.

Weyers (2003) studied some properties of double-auctions. Giraud and Tsomocos (2010) applied the double-auction mechanism to construction of dynamic monetary processes without tatonement.

6. MONEY AND SMGS

Shubik (1985) listed four properties of money: as numeraire, as means of exchange, as store of value, and as source of liquidity. Furthermore, he claimed that these properties are most naturally formalized with strategic market games.

The joint introduction of strategic market games and punishment for default has solved the traditional dichotomy between consumable goods (flow variable) and fiat money (stock variable). This approach has erased the historical divergence between microeconomics and macroeconomics analyses.

Enough money

The concept of "enough money" was introduced in Shubik (1993). It is not unrealistic that some traders may not have enough money to pay for their transactions with endogenously formed money. If traders cannot pay before a trade they need credit; if they do not have enough money after the trade to honour obligations, they default. Strategic behaviour makes players (both creditors and borrowers) consider both of these problems before trade starts.

Conditions for enough money split into three cases (Shubik, 1993):

1. Well distributed enough money - no trader has a liquidity constraint on trade.
2. Badly distributed enough money - there are traders who need liquidity and want to borrow, and there are traders who do not have the liquidity constraint and can give credit. Thus traders can organize a capital market themselves without outside intervention.

3. Not enough money in the whole economy. The economy needs the intervention of a money donor, the central bank.

Some money borrowing/lending mechanisms, based on strategic market games and endogenous demand for money, are presented below.

6.1 Endogenous demand for money

The setup of the seminal Shubik-Wilson model (1977) to introduce endogenous demand for money is a cash-in-advance model – traders borrow money from a central bank before the trade. After the trade they either pay back debts or default with penalties. The idea of their model is "money is the substitute for trust" (Shubik, in many papers) for credit/borrowing operations.

They constructed an explicit incentive compatibility mechanism for a borrower to pay his credit back or to have default with a loss in utility. Shubik and Wilson (1977) described this as follows: "If an individual ends up with a positive amount of money after having paid the bank, this has no positive value to him. If on the other hand he is unable to honour his debts in full, a penalty is levelled against him".

In their model, trade in one consumable good takes place for non-consumable fiat money. Money holding does not increase the utility of a player, but there is disutility if a borrower cannot pay back his fiat money debt.

For simplicity, traders do not have money in their endowments and take credits from the central bank. The credit market is organized as a strategic market game - traders supply their bids, and the central bank supplies money. The resulting interest rate comes from the interaction of demand for liquidity (credits or fiat money) and supply of fiat money.

The important part of the credit model is the paired appearance of two financial instruments - fiat money, issued by the central bank, and traders' promises to pay back their debts. Traders issue their promises in exchange for money obtained from credit.

Let v^1, v^2 be bids for credits from two traders[3]. The central bank is a strategic dummy in the model, which only supplies a fixed quantity of money M. The interest rate is determined in the model following the rule of strategic market games:

$$(1+r) = \frac{v^1 + v^2}{M}$$

If a trader bids for credit v^i then he will receive $\dfrac{v^i}{1+r}$ units of money in exchange for a promissory note IOU (I-Owe-You) size v^i, which is measured in fiat money. v^i is the promise of i to pay his debt back to the bank.

If there is no punishment for not repaying the credit, then the demand for money will be infinite. If the punishment is infinite (capital punishment, for example), then there will be no demand for credit money. Thus the size of the punishment affects individual trading strategies, including the demand for money, which is also a strategy in this model.

A trader has a utility maximization problem, where punishment enters as disutility[4]:

$$u^i\left(\left(b^i, q^i\right), v^i\right) - \lambda^i \max\left\{0, q^i \frac{b_1^i + b_1^j}{q_1^i + q_1^j} - b_1^i + \frac{v^i}{1+r} - v^i\right\},$$

where λ^i is the punishment for not repaying credit or a coefficient of disutility. The second term consists of monetary income from sales, monetary payments, a credit received, and a payment for the credit.

The most important result of the model is that the introduction of punishment for credits allows the demand for money to be endogenized. In other words,

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[3]   Or aggregated types of trades

[4]   There are other ways to introduce punishment

Shubik and Wilson constructed fundamentals of general equilibrium microeconomic theory with fiat money in finite time.

Shubik and Wilson (1977) used type symmetric Nash equilibrium. This means that there is a finite number of types of players (two types in their model) and there is a continuum of players of each type. In this case equilibrium in the model is competitive, which facilitates the analysis.

Shubik and Wilson (1977) did not supply a proof of existence of an equilibrium, but only a numerical example. Due to the continuum of the number of traders the optimal punishment λ^i must be equal to the marginal utility of money usage.

Using this approach, Shubik and Tsomocos (2001) constructed a playable game where a government is able to extract seigniorage from the agents in an economy, who take credits in fiat money. The government attempts to reduce the interest rate, subject to its requirement to replace worn out fiat money. Minimization of interest rates leads to minimization of the effective money supply.

The strategic variable interest rate determines revenues. "The existence of an equilibrium requires that we believe that the government can announce *in advance* the correct interest rate and how it is going to spend revenues it has not yet received" (Shubik and Tsomocos, 2001). Shubik and Tsomocos (1992) studied an economy, where a continuum of traders organize a mutual bank.

For a multiple period trade there is the important problem of how default in the past limits access to credit resources in the future. This problem was studied by Tsomocos (2007), using strategic market games.

The introduction of punishment for default in SMGs has opened a new class of general equilibrium models with fiat money in finite time, which explicitly resolves the Hahn paradox mentioned above. Fiat money does have value in a one period trade model after introduction of punishment for default.

7. APPLICATIONS OF STRATEGIC MARKET GAMES WITH MONEY

7.1 Variety of models with money

Shubik (1996) presented taxonomy of models with fiat money and endogenous demand for money. There are at least 12 basic models, which differ by sources of money, sources of uncertainty, and timing of trade. Each model has at least one feature from each of the three lists:

- Fiat money
- Outside credit
- Inside credit

- No exogenous uncertainty
- Exogenous uncertainty

- Finite horizon
- Infinite horizon

Outside credit means that there is an outside bank, which is ready to borrow and to lend. Inside credit means that traders can organize capital market themselves, using initial endowments of money. Introduction of money into general equilibrium can be done following the mechanism of Shubik and Wilson (1977). There are plenty of different variants of strategic market games, some of which will be investigated below.

Dubey and Geanakoplos (1992) proposed a way to combine inside money (loans) and outside money (money in wealth or in endowments) into the general equilibrium framework. In their model money also has value in the one-period general equilibrium model. The important outcome of their model is that these two types of money have opposite effects on interest rates in the economy. They proved existence of monetary equilibrium for a type symmetric model with continuum of traders of each type. The credit market operates in the style of Shubik and Wilson (1992).

In brief, their model consists of the following elements (we follow the notation of Dubey and Geanakoplos, 1992).

There is a set of players $H = \{1, ... h\}$ and there is a set of goods, $L = \{1, ... l\}$. A player $\alpha \in H$ has initial endowment $e^{\alpha} \in R_+^L$ and utility function $u^{\alpha} : R_+^L \rightarrow R$, concave and continuously differentiable. Restrictions $e^{\alpha} = (e_1^{\alpha}, ..., e_L^{\alpha}) \neq 0$ and $\sum_{\alpha \in H} e^{\alpha} > 0$ are necessary conditions for trade in all L goods.

Fiat money in the economy serves as a medium of transactions. Let $M > 0$ be the supply of money by the central bank and let $m^{\alpha} > 0$ be the private endowment of money of trader α. The trader can use his endowment to pay back his credit after the trade, as after the trade he does not need money. Thus after the trade the central bank accumulates all the money in the economy: M and $\sum_{\alpha \in H} m^{\alpha} > 0$.

The model has a competitive banking sector. It provides the credit and imposes penalties for borrowers in cases of default.

The game has the following order of events.

1. Each player α borrows $c^{\alpha} \in R_+$ units of money from the bank. His debt is $\mu^{\alpha} = (1 + \theta) c^{\alpha}$, where the interest rate is formed as

$$(1 + \theta) = \frac{M}{\sum_{\alpha \in H} c^{\alpha}}$$

2. All players trade with commodities using money for purchases.

The set of strategies available for trade is

$$\left(b^{\alpha},q^{\alpha}\right) \in \left\{ R_{+}^{L} \times R_{+}^{L} : \forall j \in L; q_{j}^{\alpha} \leq e_{j}^{\alpha}; \sum_{j \in L} b_{j}^{i} \leq c^{\alpha} + m^{\alpha} \equiv \frac{c^{\alpha}}{1+\theta} + m^{\alpha} \right\}$$

The final holding of fiat money is $\overline{c}^{\alpha} = c^{\alpha} + m^{\alpha} - \sum_{j \in L} b_{j}^{\alpha} + \sum_{j \in L} p_{j} q_{j}^{\alpha}$,

which consists of credit, initial endowment of money, and the monetary result of his net trade in consumable goods.

3. A player α chooses to repay $r^{\alpha} \leq \overline{c}^{\alpha}$ on his loan. His outstanding debt at the bank is $d^{\alpha} = d^{\alpha}\left(m^{\alpha}, r^{\alpha}\right) = \mu^{\alpha} - r^{\alpha}$. The choice set of player i is

$$\Sigma^{\alpha}\left(\theta, p\right) = \left\{ \begin{array}{l} \left(\mu^{\alpha}, b^{\alpha}, q^{\alpha}, r^{\alpha}\right) \in R_{+} \times R_{+}^{L} \times R_{+}^{L} \times R : q_{j}^{\alpha} \leq e_{j}^{\alpha}; \forall j \in L; \\ \sum_{j \in L} b_{j}^{\alpha} \leq \dfrac{\mu^{\alpha}}{1+\theta} + m^{\alpha}; r^{\alpha} \leq \dfrac{\mu^{\alpha}}{1+\theta} + m^{\alpha} - \sum_{j \in L} b_{j}^{\alpha} + \sum_{j \in L} p_{j} q_{j}^{\alpha} \end{array} \right\}$$

A trader cannot pay back more than his net balance in fiat money after the trade.

The outcome functions x_{j}^{α} and d_{j}^{α}, are continuous functions from $\Sigma^{\alpha}\left(\theta, p\right)$ into R_{++}.

The motivation to pay debt operates in the same way as in the standard model of bankruptcy of Shubik and Wilson (1977). If there is no punishment, then $r^{\alpha} = 0$ and there is an infinite demand for credit. Prices will be driven to infinity and the value of money will be zero. If the punishment is very high, then there is no demand for money, but the economy may have not enough money for trade.

The utility of trader α is described as:

$$U^{\alpha}\left(x^{\alpha}, d^{\alpha}\right) = u^{\alpha}\left(x^{\alpha}\right) - \lambda^{\alpha}\left(\theta, p, \omega\right) \max\left\{0, d^{\alpha}\right\},$$

where ω are other relevant macro-variables.

Monetary equilibrium is a triple (θ, p, y), $y = (y^1, \ldots, y^\alpha, \ldots, y^L)$ and $y^\alpha \in R_+^L$ and $\alpha \in H$ such that for every player $\alpha \in H$ there is: $(\mu^\alpha, b^\alpha, q^\alpha, r^\alpha,)_{\alpha \in H}$ such that

$$(\mu^\alpha, b^\alpha, q^\alpha, r^\alpha)_{\alpha \in H} \in \mathrm{argmax}_{(\mu^\alpha, b^\alpha, q^\alpha, r^\alpha) \in \Sigma^\alpha} U^\alpha \left(x^\alpha(b, q, p), d^\alpha(\mu, r), \theta, p, \omega \right)$$

$$\begin{cases} y^\alpha \equiv x^\alpha(b, q, p) \\ \omega = \eta(\theta, p, q) \\ \sum_{\alpha \in H} y^\alpha = \sum_{\alpha \in H} e^\alpha \\ \dfrac{\sum_{\alpha \in H} \mu^\alpha}{1 + \theta} = M \end{cases} ,$$

where every player maximizes his utility U^α, correctly anticipating macro-variable ω, with commodity market clearing conditions $\sum_{\alpha \in H} y^\alpha = \sum_{\alpha \in H} e^\alpha$

$\sum_{\alpha \in H} y^\alpha = \sum_{\alpha \in H} e^\alpha$ and money market clearing $\dfrac{\sum_{\alpha \in H} \mu^\alpha}{1 + \theta} = M$.

The definition of the monetary equilibrium is constructed in such a way that, if it exists, then money has positive value.

If the initial allocation is already Pareto-efficient, then there is a single trivial equilibrium, money has no value, and monetary equilibrium does not exist. If the initial endowment of fiat money in the economy $\sum_{\alpha \in H} m^\alpha > 0$ is positive, then in any monetary equilibrium there is a positive interest rate and the set of monetary equilibriums is determinate. No trader will hold money after the trade. After the trade all the money $M + \sum_{\alpha \in H} m^\alpha$ will go back to the bank.

As $\dfrac{M}{\sum_{\alpha \in H} m^{\alpha}} \to 0$, then the interest rate converges to 0 and the monetary equilibrium commodity allocations converge to the Arrow-Debreu allocations of the underlying economy without money.

The important property of the model is that it discriminates between changes in private endowments of money $\left(m^{\alpha}\right)_{\alpha \in H}$ and changes in total quantity of borrowed money M. If there is an increase in M (holding $\left(m^{\alpha}\right)_{\alpha \in H}$ constant) then the interest rate is lower, while an increase in $\left(m^{\alpha}\right)_{\alpha \in H}$ (holding M constant) raises the interest rate. This distinction is very important for studying and constructing monetary and fiscal policies.

Multiple different extensions of this model exist. Dubey and Geanakoplos (2003) showed conditions when monetary equilibrium can exist and money has positive value; even when a general equilibrium with incomplete markets may not exist.

This model has four special features in comparison to the standard uncertainty framework: "missing assets, in the sense that some imaginable contracts are not available for trade; missing market links, in the sense that not all pairs of instruments in the economy trade directly against each other; inside and outside fiat money; and a banking sector, through which agents can borrow and lend money."

7.2 Liquidity trap

A liquidity trap is when the monetary policy of the central bank loses control over the interest rate in an economy. A strategic market game is the only tool (at present) that is able to present a micro-foundation for a liquidity trap. We will demonstrate a simple example of the liquidity trap using an example from Shubik and Quint (2004). The example has several cases, which are based on outside money and the "enough money" concepts introduced above.

There are two types of traders with non-symmetric distribution of money endowments $(a, 0, m_1)$ and $(a, 0, m_2)$, where Type 1 has consumable good 1 and consumable money, and Type 2 has consumable good 2 and consumable money. m_1 and m_2 are outside money. There is a continuum of traders of each type.

There are three markets in the model - each good of the two is traded for money and there is a money market where traders Type 1 can lend money to traders Type 2.

A strategy of Type 1 is denoted by (g, q, b), where g is the total amount of money offered by this type to the money market, q - the amount of good 1 offered for sale, and b the amount of money bid for good 2. The notation for Type 2 is $(\bar{d}, \bar{q}, \bar{b})$, where $\bar{d}$ is the total demand of individual (I-O(we)-Y(ou)) bids for the commodity money.

For Type 1 lenders, the optimization problem is

$$\max_{(g,q,b)} \phi\left(a - q, \frac{b}{p}\right) + m_1 - g - b + pg + (1 + \rho)g$$

subject to

- $m_1 - g - b \geq 0$ (λ)
- $m_1 - g - b + pg \geq 0$, (μ)
- $b, g \geq 0; 0 \leq q \leq a$

The constraint $m_1 - g - b \geq 0$ means that Type 1 cannot pay for good 2 and lend more money than s/he has. The constraint $m_1 - g - b + pg \geq 0$ is the money holding of Type 1 *after* the trade. For Type 1 the second constraint follows from the first one. λ and μ are Lagrangian multipliers.

For Type 2, who are net borrowers in the model, the optimization problem is

$$\max_{(\bar{d},\bar{q},\bar{b})} \phi\left(\frac{\bar{b}}{\bar{p}}, a - \bar{q}\right) + m_1 - \frac{d}{1+\rho} - \bar{b} + p\bar{q} - \bar{d}$$

subject to

- $m_1 - \dfrac{\bar{d}}{1+\rho} - \bar{b} \geq 0$, $(\bar{\lambda})$
- $m_1 - \dfrac{\bar{d}}{1+\rho} - \bar{b} + p\bar{q} \geq 0$, $(\bar{\mu})$
- $\bar{b}, \bar{d} \geq 0; 0 \leq \bar{q} \leq a$

The constraint $m_1 - \dfrac{\bar{d}}{1+\rho} - \bar{b} \geq 0$ means that Type 2 has a positive quantity of money from his endowment and from credit. The constraint $m_1 - \dfrac{\bar{d}}{1+\rho} - \bar{b} + p\bar{q} \geq 0$ is the positive money holding of Type 2 *after* the trade.

Different from Type 1, the constraints for Type 2 are independent. If m_2 is relatively small in comparison to m_1, then Type 2 players may meet liquidity constraint and may borrow from Type 1. However, this happens only if there is not enough money in the economy and money is unequally distributed. Lagrangian multipliers λ and $\bar{\lambda}$ serve as shadow prices for money.

Continuum of traders implies perfectly competitive prices. Market clearing conditions are: for consumable goods in terms of money

$$p = \frac{\bar{b}}{q}, \bar{p} = \frac{b}{\bar{q}}$$

and for the money market the interest rate is determined as

$$1+\rho=\frac{\overline{d}}{g}$$

Shubik and Quint (2004, p.13) presented sensitivity analysis for the model. Table 1 contains some of their results (from Shubik and Quint, 2004, p.13).

Table 1. Prices from the example of a liquidity trap

m_1+m_2	m_2	$\dfrac{m_2}{m_1+m_2}$	p	$\overline{p}$	$\overline{\rho}$	$\overline{\lambda}$	$\overline{\mu}$
$m_1+m_2>a$	$<m_1$	any	1	?	0	0	0
$m_1+m_2=a$	$<m_1$	any	1	1	0	0	0
$m_1+m_2=a/5$	7a/8	7a/16	½	½	3	3	3
$m_1+m_2=a/5$	0	0	0.73	0.32	3.25	3.25	3.25

(**Source:** Shubik and Quint, 2004)

There are four cases in the analysis of this simple model. The complete analysis of the model is presented in Appendix B of Shubik and Quint (2004).

Case 1. $m_1+m_2>a$. There is enough money in the economy and the shadow price of money is zero, $\lambda=\overline{\lambda}=0$. Type 2 pays a zero interest rate and can demand any quantity. $\rho=0$ is the interest rate (price of the transaction service of the consumable money). This is the case of a liquidity trap. If there is an increase in the total quantity of money (in the players' endowments), this will have some effect on the interest rate.

Case 2. $m_1+m_2=a$. There is exactly enough money to serve the trade in the economy.

The common feature between cases 1 and 2 is that traders do not need to organize a credit market.

Case 3. $0 < m_1 + m_2 < a$ and $m_1 \approx m_2$, $m_2 < m_1$. The shadow price of the transaction value of the money is positive and traders open a money (credit) market. The money market reallocates resources as a standard capital market.

The approximate interest rate for this case is $\rho \approx \dfrac{a}{\sqrt{2(m_1 + m_2 + 2)}} - 1$.

Case 4. $0 < m_1 + m_2 < a$ and $m_1 << m_2$. For example, $m_1 = a/5$, $m_2 = 0$. Money is highly non-symmetrically distributed in the economy and there is not enough money in the economy. Type 2 is constrained by bankruptcy conditions.

In cases 3 and 4 traders organize a credit market and the money supply can affect interest rates.

Shubik and Quint (2004) concluded their example with the words:

"... as long as there is some outside money in the system there are many ways in which one can construct one-period economies, which avoid the Hahn paradox of no trade. In particular here it is avoided by the use of consumable money, which thereby maintains its value at the terminal point of the game."

7.3 Other macroeconomic applications of monetary equilibrium models with default

The paper of Tsomocos et al. (2003) has multiple features important for applied macroeconomic analysis: incomplete markets with money and default, a non-bank private sector, banks, a central bank, a government, and a regulator. The model has positive default in equilibrium. The model characterises "contagion and financial fragility as an equilibrium phenomenon, based on individual rationality".

Another important paper is by Goodhart, Sunirand and Tsomocos (2006), who studied rationality of bank behaviour, possible contagious default, approaches to designing prudential regulation, and approaches to limiting incentives for excessive risk-taking by banks. The applied side of this paper is that it supplies

micro-foundations for financial fragility mechanisms, and "highlights the trade-off between financial stability and economic efficiency".

Tsomocos et al. (2007) studied the multi-period inter-bank credit market. The authors demonstrated how banks become inter-connected through promissory notes, which results in dynamic spill-over effects (negative externalities for the banking industry): the financial results of one bank sequentially affect profits and default rates of another.

Tsomocos (2008) investigated nominal indeterminacy in a monetary overlapping generation model of the international economy. He demonstrated that the combined effect of the monetary sector together with the market and agent heterogeneity remove real and nominal indeterminacy. The important partial result is that existence of outside money removes the nominal indeterminacy. The resulting "monetary policy becomes non-neutral since monetary changes affect nominal variables which in turn determine different real allocations".

Karatzas, Shubik and Sudderth (2008) applied SMGs to studying fiscal and monetary control and government decisions.

Geanakoplos and Tsomocos (2002) studied applications for international finance, while Peiris and Tsomocos (2009) studied international monetary equilibrium with default. The application of SMGs to studying monetary economics within a general equilibrium framework is a growing field of research.

8. EXPERIMENTS WITH SMGS

There is some literature on experiments with strategic market games. Duffy, Matros and Tezemtelides (2009) investigated convergence of SMGs with competitive outcome. They reported that as the number of participants increases, the Nash equilibrium they achieve approximates the associated Walrasian equilibrium of the underlying economy.

Huber, Shubik and Sunder (2007) made experiments with endogenous demand for money. Players issued their own IOU promises. Settlement was done by a costless efficient clearing-house. Their results suggest that "if the information system and clearing are so good as to preclude moral hazard and any form of information asymmetry, then the economy operates efficiently at any price level without government money".

Anger et al. (2009) conducted similar experiments, and asked the following research question: "Is personal currency issued by participants sufficient for an economy to operate efficiently, with no outside or government money?" The results demonstrate that "if agents have the option of not delivering on their promises, a high enough penalty for non-delivery is necessary to ensure an efficient market; a lower penalty leads to inefficient, even collapsing, markets due to moral hazard."

Huber, Shubik and Sunder (2011) investigated the applicability of penalties for equilibrium selection in SMGs. They report experimental evidence on the effectiveness of penalties for conversion to a desired equilibrium.

The results of the last three papers are very close to the predictions of the Prisoners' Dilemma in terms of monetary economics. This prediction can be formulated in the words of Shubik: "money is a substitution for trust". It may be individually rational not to repay credit, but this closes credit markets as it destroys the credibility of borrowing. In order to avoid it there has to be credible punishment imposed on the players. Shubik's prediction is very relevant to many debt crises and the (re-)construction of monetary unions.

In another paper, Huber, Shubik and Sunder (2010) reported the results of experiments where they compared predictions of SMG theory with the results of different experimental strategic market games (sell all, buy-sell, and double auction). Their data reveal different paths of convergence and different levels of allocative efficiency in the three settings. These results suggest that institutional details matter in understanding differences between the investigated games.

9. CONCLUSION

This small survey has highlighted some of the features of SMG and some current applications of SMG for macroeconomic analysis.

REFERENCE

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Amir, R., Bloch, F. (2009). Comparative statics in a simple class of strategic market games. *Games and Economic Behavior,* 65 (1), pp. 7–24.

Amir, R., Sahi, S., Shubik, M. & Yao, S. (1990). A strategic market game with complete markets. *Journal of Economic Theory,* 51 (1), pp. 126–143.

Angerer, M., Huber, J., Shubik, M. & Sunder, S. (2010). An economy with personal currency: theory and experimental evidence, *Annals of Finance*, 6 (4), pp. 475-509.

Goodhart, C., Sunirand, P., Bhattacharya, S. & Tsomocos D. (2007). Banks, relative performance, and sequential contagion. *Economic Theory,* 32 (2), pp. 381–398.

Bloch, F., Ferrer, H. (2001). Trade fragmentation and coordination in strategic market games. *Journal of Economic Theory,* 101 (1), pp. 301–316.

Brandenburger A. (2008). Epistemic game theory: an overview. In *The New Palgrave Dictionary of Economics*. Palgrave Macmillan, Basingstoke

Brangewitz, S. (2012). A strategic market game with a finite horizon, incomplete markets and the possibility of default using collaterals, PhD dissertation. Bellefeld, Germany.

Busetto, F., Codognato, G. (2006). Very nice trivial equilibria in strategic market games. *Journal of Economic Theory,* 131 (1), pp. 295–301.

Capie, F. H., Tsomocos, D. P. & Wood G. E. (2006). Modeling institutional change in the payments system, and its implications for monetary policy? In Schmitz, S. W. & Wood, G. (eds.) *Institutional change in the payments system and monetary policy* (pp. 62-80). Abingdon, UK: Routledge.

Jackson Catarineu-Rabell, E., Tsomocos, D. P. (2003). *Equilibrium analysis, banking, contagion and financial fragility*. Financial Markets Research, London School of Economics.

Cordella, T., Gabszewicz, J.J. (1998). Nice trivial equilibria in strategic market games. *Games and Economic Behavior*, 22 (1), pp. 162–169.

Dubey, P. (1980). Nash equilibria of market games: Finiteness and inefficiency. *Journal of Economic Theory,* 22 (2), pp. 363–376.

Dubey, P. (1982). Price-quantity strategic market games. *Econometrica: Journal of the Econometric Society,* pp. 111–126.

Dubey, P., Geanakoplos, J. (2003). From Nash to Walras via Shapley-Shubik. *Journal of Mathematical Economics,* 39 (5-6), pp. 391–400.

Dubey, P., Geanakoplos, J. (2003). Monetary equilibrium with missing markets. *Journal of Mathematical Economics,* 39 (5-6), pp. 585–618.

 Dubey, P., Geanakoplos, J. & Shubik, M. (1987). The revelation of information in strategic market games: A critique of rational expectations equilibrium. *Journal of Mathematical Economics,* 16 (2), pp.105–137.

Dubey, P., Geanakoplos, J. (1992). The value of money in a finite horizon economy: a role for banks. *Economic Analysis of Market Games, Essays in Honor of Frank Hahn*, pp. 407–444.

Dubey, P., Mas-Colell A. & Shubik M. (1980). Efficiency properties of strategies market games: An axiomatic approach. *Journal of Economic Theory,* 22 (2), pp. 339–362.

Dubey, P., Sahi, S. (2003). Price-mediated trade with quantity signals: an axiomatic approach. *Journal of Mathematical Economics*, 39 (5-6), pp. 377–389.

Dubey, P. and Shubik, M. (1977). Trade and prices in a closed economy with exogeneous uncertainty, different levels of information, money and compound futures markets. *Econometrica: Journal of the Econometric Society,* pp. 1657-1680.

Dubey, P. and Shubik, M. (1980). A strategic market game with price and quantity strategies. *Journal of Economics,* 40 (1), pp. 25–34.

Dubey, P. and Shubik, M. (1978). *A Theory of Money and Financial Institutions. Part 37. The Profit Maximizing Firm: Managers and Stockholders*. (Cowles Foundation Discussion Paper No. 483), Yale University, New Haven.

Dubey, P., Shubik, M. (1977). A closed economic system with prqduction and exchange modeled as a game of strategy. *Journal of Mathematical Economics,* 4, pp. 253–287.

Dubey, P., Sondermann, D. (2009). Perfect competition in an oligopoly (including bilateral monopoly). *Games and Economic Behavior,* 65 (1), pp. 124–141.

Duffy, J. and Matros, A. & Temzelides, T. (2011). Competitive behavior in market games: Evidence and theory. *Journal of Economic Theory,* 46, pp. 1437-1463.

Geanakoplos, J., Tsomocos D. (2002). International finance in general equilibrium. *Research in Economics* , 56 (1), pp. 85–142.

Ghosal, S., Morelli, M. (2004). Retrading in market games. *Journal of Economic Theory,* 115 (1), pp. 151–181.

Giraud, G. (2003). Strategic market games: an introduction. *Journal of Mathematical Economics,* 39 (5-6), pp. 355–375.

Giraud, G., Stahn, H. (2008). On Shapley–Shubik equilibria with financial markets. *Economic Theory*, 35 (3), pp. 469–496.

Giraud, G., Tsomocos D. (2010). Nominal uniqueness and money non-neutrality in the limit-price exchange process. *Economic Theory,* 45 (1), pp. 303–348.

Giraud, G., Weyers, S. (2004). Strategic market games with a finite horizon and incomplete markets. *Economic Theory,* 23 (3), pp.467–491.

Goenka, A., (2003). Informed trading and the leakages of information. *Journal of Economic Theory,* 109 (2), pp. 360–377.

Goodhart, C. A. E., Sunirand, P. & Tsomocos, D. P. (2006). A model to analyse financial fragility. *Economic Theory,* 27 (1), pp.107–142.

Gottardi, P., Serrano, R. (2005). Market power and information revelation in dynamic trading. *Journal of the European Economic Association*, 3 (6), pp. 1279–1317.

Hens, T., Reimann, S. & Vogt, B. (2004). Nash competitive equilibria and two-period fund separation. *Journal of Mathematical Economics,* 40 (3-4), pp. 321–346.

Huber, J., Shubik, M. & Sunder, S. (2007). *Everyone-a-banker or the ideal credit acceptance game: Theory and evidence*. Levine's Bibliography. Electronic Library. UCLA Department of Economics, California, USA.

Huber, J., Shubik, M. & Sunder, S. (2009). *Default Penalty as Disciplinary and Selection Mechanism in Presence of Multiple Equilibria*. (Cowles Foundation Discussion Paper No. 1730). Yale University, New Haven.

Huber, J., Shubik, M. & Sunder, S. (2010). Three minimal market institutions with human and algorithmic agents: Theory and experimental evidence. *Games and Economic Behavior,* 70 (2), pp. 403-424.

Huber, J., Shubik, M. & Sunder, S. (2011). *Default penalty as a selection mechanism among multiple equilibria*. (Cowles Foundation Discussion Paper No. 1730R), Yale University, New Haven.

Karatzas, I., Shubik, M. & Sudderth W. (2008). *Financial control of a competitive economy without randomness*. (Cowles Foundation Discussion Paper No. 1681), Yale University, New Haven.

Kiyotaki, N., Wright, R. (1989). On money as a medium of exchange. *The Journal of Political Economy* , pp. 927–954.

Koutsougeras, L. C. (2003). Non-Walrasian equilibria and the law of one price. *Journal of Economic Theory,* 108 (1), pp. 169–175.

Koutsougeras, L. C. (2009). Convergence of strategic behavior to price taking. *Games and economic behavior,* 65 (1), pp. 234–241.

Koutsougeras, L. C. (1999). *A remark on the number of trading posts in strategic market games*. (Discussion paper of CORE No. 99005), Universite catholique de Louvain, Belgia.

Koutsougeras, L., Papadopoulos, K. (2004). Arbitrage and equilibrium in strategic security markets. *Economic Theory,* 23 (3), pp. 553–568.

Koutsougeras, L. C., Ziros, N. (2008). A three way equivalence. *Journal of Economic Theory,* 139 (1), pp. 380–391.

Levando, D., Boulatov, A., & Tsomocos, D. (2012). Epistemic mixed strategies in strategic market games with wash sales, (*mimeo*).

Mertens, J. F. (2003). The limit-price mechanism. *Journal of Mathematical Economics,* 39 (5-6), pp. 433–528.

Minelli, E., Meier, M., (2011). A large market game with private information on both sides. *(mimeo)*.

Papadopoulos, K. G., (2008). Purchasing power parity with strategic markets. *The BE Journal of Theoretical Economics*, 8 (1), Berkeley Electronic Press.

Papadopoulos, K. G., Koutsougeras. L. C. (2011). A strategic market game of carry trades and equilibrium profits. *The Manchester School,* 79 (3), pp. 528–541.

Peck J., Shell, K., (1989). On the nonequivalence of the arrow-securities game and the contingent-commodities game, Part 1, in W. Barnett, J. Geweke and K. Shell (Eds.), *Economic complexity: Chaos, sunspots, bubbles, and nonlinearity* (pp. 61–85). Cambridge University Press, New York.

Peck J., Shell, K. (1990). Liquid markets and competition. *Games and Economic Behavior,* 2 (4), pp. 362–377.

Peck, J., Shell, K., & Spear, S. (1992). The market game: existence and structure of equilibrium. *Journal of Mathematical Economics,* 21 (3), pp. 271–99.

Peiris, U., Tsomocos, D. (2009). *International monetary equilibrium with default*. (Saıd Business School WP, April, 9).

Polemarchakis, H. M., Ray, I. (2006). Sunspots, correlation and competition. *Games and Economic Behavior,* 56 (1), pp. 174–184.

Quint, T., Shubik, M. (2004). *Gold, Fiat and Credit: An Elementary Discussion of Commodity Money, Fiat Money and Credit.* (Cowles Foundation Discussion Paper No. 1460), Yale University, New Haven.

Ray, I. (2001a). Buying and selling in strategic market games. *Economics Letters,* 71 (1), pp. 49–53.

Ray, I. (2001b). On games with identical equilibrium payoffs. E*conomic Theory,* 17 (1), pp. 223–231.

Rogawski J., Shubik, M. (1986). A strategic market game with transactions costs. *Mathematical Social Sciences,* 11 (2), pp. 139–160.

Shapley. L., Shubik, M. (1977). Trade using one commodity as a means of payment. *The Journal of Political Economy,* 85 (5), pp. 937–968.

Shubik, M. (1959). Strategy and market structure: competition, oligopoly, and the theory of games. *(mimeo).* Yale University.

Shubik, M. (1985a). A note on enough money in a strategic market game with complete or fewer markets. *Economics Letters,* 19 (3), pp. 231–235.

Shubik, M. (1985b). *The Many Properties of Money: A Strategic Market Game Analysis.* (Cowles Foundation Discussion Paper No. 759), Yale University, New Haven.

Shubik, M. (1990). A game theoretic approach to the theory of money and financial institutions. *Handbook of Monetary Economics*, v.1. (pp. 171–219). Elsevier.

Shubik, M. (1990). The transactions trust demand for money. *Journal of Economics,* 52 (3), pp. 211–232.

Shubik, M. (2004). *The theory of money and financial institutions*, v. 1. MIT Press.

Shubik, M., Tsomocos, D. P. (1992). A strategic market game with a mutual bank with fractional reserves and redemption in gold. *Journal of Economics,* 55 (2), pp. 123–150.

Shubik, M., Tsomocos, D. P. (2001). A strategic market game with seignorage costs of Fiat money. *Economic Theory,* 19 (1), pp. 187–201.

Shubik, M., Wilson, C. (1977). The optimal bankruptcy rule in a trading economy using fiat money. *Journal of Economics,* 37 (3), pp. 337–354.

Sorin, S. (1996). Strategic market games with exchange rates. *Journal of Economic Theory,* 69 (2), pp. 431–446.

Tsomocos, D. P. (2003). Equilibrium analysis, banking and financial instability. *Journal of Mathematical Economics,* 39 (5-6), pp. 619–655.

Tsomocos, D. P. (2008). Generic determinacy and money non-neutrality of international monetary equilibria. *Journal of Mathematical Economics,* 44 (7-8), pp. 866–887.

Tsomocos, D. P. Bhattacharya, S. & Goodhart, C. A. E., Sunirand, P. (2007). Banks, relative performance, and sequential contagion. *Economic Theory*, 32 (2). pp. 381-398.

Weyers, S. (1999). Uncertainty and insurance in strategic market games. *Economic Theory,* 14 (1), pp.181-201.

Weyers, S. (2003). A strategic market game with limit prices. *Journal of Mathematical Economics,* 39 (5-6), pp. 529–558.