An alternative approach to portfolio management

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Abstract - This work continues author's investigations about using multistage (or continuous) version of VaR-criterion in portfolio management. In the market of quite arbitrary nature, principles of constructing the optimal portfolio of an investor with own view on market properties and own risk preferences are established. These principles allow transgressing the bounds of both theory of second order of Markowitz in portfolio management and the standard VaR-criterion that is often used in problems of portfolio hedging.

A scenario approach is applied. In the space of all possible random combinations of some factors that operate in the market and generate the random return of all securities under consideration, a finite subset is chosen, which can be considered as a representative sample of the original factor space. The cardinality of this subset will determine the precision of the investor's problem solution and must be coordinated with the multiplicity of market instruments.

Some examples are considered that demonstrate the efficiency of the method developed. It is shown how uniformly the problems of managing an arbitrary securities portfolio and hedging can be described.

I. THEORETICAL ONE-PERIOD MARKET

This work continues author's investigations presented in [1,2]. An arbitrary one-period market with N elementary securities $\hat{s}_i$, $i \in I = \{1, 2, \ldots, N\}$, is considered. We mark out these securities among all possible securities for convenience meaning two considerations: (1) these are by implication the simplest securities in the market, and (2) all other instruments and only they can be represented as linear combinations of these elementary securities. We below use vector and matrix notifications, so we denote a set of all $\hat{s}_i$, $i \in I$, as a vector $\hat{\boldsymbol{s}}$. Their prices are $m_i = |\hat{s}_i|$, $i \in I$. In the market, there operate some factors, which affect the actual random return on securities. All their combinations form a space Ω_0. We choose in this space a finite subset $\Omega \subset \Omega_0$, which can be considered as a representative sample of the original set Ω_0. All these sampling points will be referred to as scenarios. We call for $\Omega = I$. A single long position in i-th instrument $\hat{s}_i$ generates the random return y_i. We denote the return on instruments $\hat{s}_i$, $i \in I$, under scenarios $j \in \Omega$, as $\hat{s}_i(j) = y_{ij}$; they form a matrix $\boldsymbol{Y}$.

Let's introduce new instruments. We call $\hat{u}_i$, $i \in I$, *basic* instruments if $\hat{u}_i(j) = \delta_{ij}$, $i \in I$, where δ_{ij} is a Kroneker symbol. If we replicate $\hat{\boldsymbol{u}}$ by $\hat{\boldsymbol{s}}$ in the form $\hat{u}_i = \sum_{j \in I} z_{ij} \hat{s}_j$, or $\hat{\boldsymbol{u}} = \boldsymbol{Z}\hat{\boldsymbol{s}}$, we readily derive that $\boldsymbol{Z} = \boldsymbol{Y}^{-1}$ (assuming the existence of the matrix $\boldsymbol{Y}^{-1}$), and so $\hat{\boldsymbol{u}} = \boldsymbol{Y}^{-1}\hat{\boldsymbol{s}}$. If we neglect all transaction costs we get $\boldsymbol{c}^T = \boldsymbol{Y}^{-1}\boldsymbol{m}^T$, where $c_i = |\hat{u}_i|$, $i \in I$, are prices of basic instruments, and superscript T denotes the operation of transposition.

We can attach to these prices a probability sense by multiplying they all by normalizing factor $r_{rf} = 1/\sum_j c_j$. The point is that the sum of all basic instruments forms a single risk-free instrument. So the factor r_{rf} determines the risk-free return relative that can be received in the market (typically $r_{rf} > 1$) and $\boldsymbol{c}° = r_{rf}\, \boldsymbol{c}$ can be interpreted as implied scenario probability vector.

Next we describe an arbitrary investor, which has own market property forecast and own risk preferences. Investor's forecast is given by a vector $\boldsymbol{d} = (d_1, d_2, \ldots, d_n)$ of scenario probabilities, which needs not coincide with vector $\boldsymbol{c}°$. Investor's risk preferences are given by a continuous monotone non-decreasing critical return function $B_{cr}(\varepsilon)$, $\varepsilon \in [0, 1]$. We require that $P_t\{R < B_{cr}(\varepsilon)\} \le \varepsilon$ for all $\varepsilon \in [0, 1]$, where $P_t\{E\}$ is a probability measure of the event E as forecasted by our investor, and R is the random return on the investment.

To solve this problem we use Neuman-Pearson method [3]. A likelihood ratio vector $\boldsymbol{l} = (l_1, l_2, \ldots, l_n)$, where $l_j = c_j/d_j$, is introduced and all scenarios from I are reordered in diminishing order of these ratios. We denote by $\xi = (\xi(1), \xi(2), \ldots, \xi(n))$ a one-one mapping of I onto itself corresponding to the above order. This mapping can be also given by a substitution matrix $\Xi = \|\xi_{ij}\|$, for which $\xi_{ij} = 1$ if $j = \xi(i)$ and $\xi_{ij} = 0$ otherwise, $i,j \in I$.

Now, with the help of this matrix, we can find the reordered vector $\boldsymbol{d_{ro}}^{\boldsymbol{T}} = \Xi \boldsymbol{d}^{\boldsymbol{T}}$. Then we determine vector $\boldsymbol{\varepsilon}$ by the rule: $\varepsilon_k = P_t\{X_k\} = \sum_{j \le k} d_{\xi(j)}$, where $X_k = (\xi(1), \xi(2), \ldots, \xi(k))$. Also we find investor's risk preference vector $\boldsymbol{b} = (b_1, b_2, \ldots, b_n)$, where $b_k = B_{cr}(\varepsilon_k)$.

This information suffices to derive investor's optimal portfolio $\hat{g}$ and its properties. It holds

$$\hat{g} = \sum_j b_j\, \hat{u}_{\xi(j)} = \boldsymbol{b}\, \Xi\, \hat{\boldsymbol{u}}^{\boldsymbol{T}} = \boldsymbol{g}\, \hat{\boldsymbol{s}}^T, \text{ where } \boldsymbol{g} = \boldsymbol{b}\, \Xi\, \boldsymbol{Y}^{-1}.$$

The value of this optimal portfolio (i.e. the investment amount) is

$$|\hat{g}| = \boldsymbol{b}\, \Xi\, \boldsymbol{Y}^{-1}\, \boldsymbol{m}^T = \boldsymbol{g}\, \boldsymbol{m}^T.$$

Optimal average return of the investor is

$$R_{opt} = \boldsymbol{b}\ \Xi\ \boldsymbol{d}^T,$$

and, hence, optimal average return relative is

$$r_{opt} = R_{opt}/|\hat{g}| = (\boldsymbol{b}\ \Xi\ \boldsymbol{d}^T)/(\boldsymbol{g}\ \boldsymbol{m}^T),$$

which must be greater than r_{rf}.

Consider a conditional example of a market with 5 securities. Let

$$\boldsymbol{Y} = \begin{pmatrix} 2.0 & 1.4 & 0.2 & 0.2 & 0.2 \\ 0.9 & 1.4 & 0.6 & 0.4 & 0.2 \\ 0.4 & 0.6 & 0.8 & 0.5 & 0.3 \\ 0.3 & 0.2 & 0.3 & 0.7 & 0.5 \\ 0.6 & 0.4 & 0.6 & 0.9 & 1.5 \end{pmatrix},$$

$$\boldsymbol{m} = (0.772, 0.707, 0.514, 0.368, 0.716).$$

According to the above procedure, we derive the vector of prices of all basic instruments

$$\boldsymbol{c} = (0.17, 0.23, 0.21, 0.19, 0.15).$$

Normalizing this vector gives the risk-free return relative equal to 1.05263. Let the investor's forecast of scenarios probabilities be given by the vector

$$\boldsymbol{d} = (0.19, 0.20, 0.23, 0.23, 0.15),$$

and investor's risk preferences be given by an increasing function of critical incomes

$$B_{cr}(\varepsilon) = \varepsilon - 0.2, \quad \varepsilon \in [0,1].$$

Applying the Neuman-Pearson procedure, we find that the likelihood ratio vector in example under consideration is

$$\boldsymbol{l} = (0.895, 1.150, 0.913, 0.826, 1.000)$$

and, therefore,

$$\xi = (2, 5, 3, 1, 4).$$

It follows that

$$\boldsymbol{d}_{ro} = (0.20, 0.15, 0.23, 0.19, 0.23),$$

$$\varepsilon = (0.20, 0.35, 0.58, 0.77, 1.00),$$

$$\boldsymbol{b} = (0.00, 0.15, 0.38, 0.57, 0.80).$$

From this information, we already can construct the investor's optimal portfolio. We derive that weighting factors of this portfolio form the vector

$$\boldsymbol{g} = (0.424, -0.843, 0.799, 1.641, -0.551).$$

The value of this optimal portfolio is $|\hat{g}| = 0.3512$, the average income of the investor is (from investor's point of view) $R_{opt} = 0.4022$. Hence, the return relative of the optimal portfolio of the investor equals to

$$r_{opt} = R_{opt}/|\hat{g}| = 1.14522,$$

which is more than r_{rf}.

II. APPLICATION TO THE ROULETTE GAME

We wish to apply the above approach to a simplified version of roulette game. We assume that the roulette has only 36 cells and does not have a 'zero' cell. The player has own view on roulette properties. It is readily seen that elementary instruments can in this example be identified with basic ones. As such, we choose all stakes, each of which is the stake of the player on one single cell of all 36 cells.

Let's suppose that, in contrast to casino, the player thinks that the probability of the roulette halt is the largest for 13th cell and equals 1.09/36, and is the least for opposite 31st cell and equals 0.91/36. In intermediate cells, the probability varies linearly. Also, we take the risk preference function of the player in the form $B_{cr}(\varepsilon) = \varepsilon^\gamma$, $\varepsilon \in [0,1]$. It is obvious that $r_{rf} = 1.00$ for all $\gamma > 0$.

If $\gamma = 4$ (i.e., the player is very much inclined to the risk) the use of Neuman-Pearson procedure provides the optimal portfolio (i.e., combined stake), for which the largest weight falls on 13th cell and the lowest weight falls on 31st cell. Also, the scattering of weights is significant. The financial properties of this portfolio are as follows:

$$|\hat{g}| = 0.20, \quad R_{opt} = 0.21, \quad r_{opt} = 1.06.$$

If $\gamma = 0.25$ (i.e. the player is averse to the risk to great extent) then, analogously, the largest weight of the optimal portfolio falls on 13th cell and the lowest weight falls on 31st cell. But this time, the scattering of weights is very small. Therefore financial results of this portfolio are much more modest. This is testified by comparison of average returns relative in both cases:

$$|\hat{g}| = 0.80, \quad R_{opt} = 0.81, \quad r_{opt} = 1.01.$$

III. ONE-PERIOD OPTION MARKET

Let's consider a one-period option market and an investor, which has own view on market behavior in the future and own risk preferences in the form of function $B_{cr}(\varepsilon) = \varepsilon^\gamma$, $\varepsilon \in [0,1]$. It is reasonable that we take the underlier's price as the only factor affecting the prices of all options. For simplicity, we assume that, in the market, 10 call options trade with the successive strikes $a + 0.1$, $a + 0.2$, …, $a + 1.0$, where a is some reference price. As it was agreed, we have to choose a sample of exactly 10 values – scenarios - of underlier's prices. As such we take the same sequence but shifted to the right of 0.1.

Recalling the payoff profile of the call option we obtain that the first line of the matrix $\boldsymbol{Y}$ is the vector (0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0).The second line of $\boldsymbol{Y}$ is derived from this line by shifting it of one position to the right, throwing away the rightmost number and putting a zero in the first position. And so forth, until the last line.

Suppose that

$\boldsymbol{m}$ = (0.52, 0.43, 0.34, 0.27, 0.20, 0.14, 0.09, 0.06, 0.03, 0.01).

and consider the first case when the investor estimates the market as less volatile than it is evidenced by call prices. Therefore we can set, for example,

$$\boldsymbol{d} = (8, 9, 10, 11, 12, 11, 10, 9, 8, 7)/95.$$

If investor's critical return function is $B_{cr}(\varepsilon) = \varepsilon^4$, $\varepsilon\in[0,1]$, we derive by applying standard Neuman-Pearson procedure the optimal portfolio in the form (here $r_{rf} = 1.05$)

$\boldsymbol{g}$ = (0.6, 12, 58, 177, 423, –1087, 5, 277, 104, 28)/100.

Its parameters are

$$|\hat{g}| = 0.21, \quad R_{opt} = 0.26, \quad r_{opt} = 1.25.$$

If, however, the investor with the same risk preferences supposes that the market volatility will increase and so, for example,

$$\boldsymbol{d} = (11, 10, 9, 8, 7, 8, 9, 10, 11, 12)/95$$

(i.e., the market and the investor simply exchange their roles in comparison with the first case), the vectors $\boldsymbol{\varepsilon}$ and $\boldsymbol{b}$ remain the same as previously and all financial characteristics of the optimal portfolio are the same too (here again $r_{rf} = 1.05$):

$$\hat{g}| = 0.21, \quad R_{opt} = 0.26, \quad r_{opt} = 1.25.$$

Nevertheless, the optimal portfolio itself, which is given in this case by the vector

$\boldsymbol{g}$ = (583, –995, 277, 104, 28, 4, 12, 58, 177, 423)/100,

undergoes a very thorough change.

IV. HEDGING IN OPTION MARKET

As supplement to this example, let us now address the *hedge problem*. Formally, the distinction of this problem from one that was just considered becomes apparent when investor chooses a special type of the critical return function. If, for example, the investor wishes to eliminate the downside risk of underlier's price reduction, he or she can add to the original function, say $B_{cr}(\varepsilon) = \varepsilon^\gamma$, the positive constant h, which determine the minimal return for investor. Letting the market and investor's forecast be given as formerly and investor's risk preferences be given by the function $B_{cr}(\varepsilon) = \varepsilon^4+h$, $\varepsilon\in[0,1]$, we could find optimal portfolios for various h by the same way as we did it above in case $h = 0$.

The hedge problem, however, is stated usually slightly otherwise. The hedger specifies some $\alpha < 1$ and tries to find such a value h to be $|\hat{g}| = \alpha h$. To solve this problem we need to organize a search of desired h, at which this relation is valid. We carry out a series of computations for prescribed α and for a set of values of h, as we did it above.

Let, for example, $\alpha = 0.8$, i.e. our investor's desire is not to lose more than 20% of the investment amount. Setting 0.5 and 1.0 as start values for h and using Newton approach together with dichotomy considerations we can quickly find out an appropriate value of h with the precision of, say, 0.01. As it can be ascertained, our investor's hedge problem is solved by the value $h = 0.71$. At that, $|\hat{g}| = 0.887$, $r_{opt} = 1.097$. Comparing this average return relative (1.10) to the average return relative without hedge (1.25) shows that such hedging comes for the investor to substantial decline in average yield.

V. BOND MARKET

Let's consider now the very simple one-year two-period bond market with 2 bonds, the coupon rates of which are 4% and 16%, respectively. Today's semiannual rate is known and equal to $\rho_0 = 5\%$. According to our model, the investor chooses in the set of all possible semiannual forward rates in a half of the year merely two values, i.e. two scenarios, – 4% and 6%, and the investor attributes to them some probabilities. The actual income on each bond is random and is determined by the forward rate, or scenario.

So, we have $I = \{1, 2\}$, and for the bond $i\in I$ with the face value F_i, the coupon C_i and forward rate ρ_j the income at the end of the period 2 will be (the first coupon payment is reinvested on the forward rate ρ_j)

$$y_{ij} = C_i\,(1+\rho_j) + C_i + F_i.$$

Computing these values, we derive

$$\boldsymbol{Y} = \begin{pmatrix} 104.08 & 104.12 \\ 116.32 & 116.48 \end{pmatrix}.$$

Next, let the market evaluate the forward rate as ρ_0, i.e., at today's level. So,

$$m_i = C_i/(1+\rho_0) + (C_i + F_i.)/(1+\rho_0)^2$$

and, by simple calculation,

$$\boldsymbol{m} = (94.422, 105.578), \quad \boldsymbol{c} = (0.4535, 0.4535).$$

Let investor's forecast of scenarios probabilities be

$$\boldsymbol{d} = (0.4, 0.6).$$

Consequently,

$$\boldsymbol{l} = (1.1338,\ 0.7559),$$

and, therefore, the mapping ξ of the set I (in its original order) onto itself is identical, and Ξ is the identity matrix. Suppose that the investor's risk preferences are given by the function

$$B_{cr}(\varepsilon) = \varepsilon^2, \quad \varepsilon \in [0,1].$$

Then

$$\varepsilon = (0.4,\ 1.0), \qquad \boldsymbol{b} = (0{,}16,\ 1.00).$$

We now obtain the investor's optimal portfolio and its financial characteristics:

$$\boldsymbol{g} = (-8.140,\ 7.285),$$

$$|\hat{g}| = 0.526077, \qquad R_{opt} = 0.664, \qquad r_{opt} = 1.26217$$

(in this case $r_{rf} = 1.1025$).

If the investor's prediction of the scenarios probabilities is different, namely

$$\boldsymbol{d} = (0.6,\ 0.4),$$

the optimal portfolio is

$$\boldsymbol{g} = (8.156,\ -7.289),$$

i.e., long and short positions in the portfolio change places. All other financial characteristics – $|\hat{g}|$, R_{opt} and r_{opt} – remain the same.

VI. CONCLUSIONS

To describe the risk preferences of portfolio investors more precisely, the method of multistage (or continuous, in some theoretical schemes) VaR-criterion, developed by the author in his previous investigations for one-period option markets, is applied to markets of quite arbitrary nature. Principles of constructing the optimal portfolio of an investor are based on investor's own forecast about joint probability distribution of market securities and risk preferences that are given in the form of an increasing function of critical incomes. Computing the optimal portfolio of the investor is provided by using the Neuman-Pearson procedure. Although this approach is approximate, it can be examined as an alternative to such methods of portfolio management as Markowitz theory and, too, the standard VaR-criterion that is often used in problems of portfolio hedging.

A scenario approach applied, with each scenario being some combination of some factors that determine the market movement, allows transforming the problem to a discrete one and controlling the precision of problem solution by chousing its cardinality. For this purpose, the number of scenarios must be coordinated with the multiplicity of the market under consideration and be equal to the number of the so-called elementary instruments.

A conditional example of a market for demonstration purposes is considered that shows the algorithm performance. Besides, the examples of some real markets including the roulette game and option and bond markets are investigated that demonstrate the efficiency of the method developed. It is shown how the problem of hedging in option market fits the description of managing a portfolio of arbitrary securities.

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